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The effect of laminar viscosity on the solution of the undular bore

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In this paper we look at the problem of an undular bore entering still water. The effect of the boundary-layer at the channel bottom is considered. The motion is assumed to be laminar and inviscid, apart from the thin boundary layer, where the normal boundary-layer approximations are used to find the velocity components. The effect on the main flow then appears in the equations of motion as a shear stress.

1. Introduction

The classical theory of the bore (Rayleigh 1914) is based on a transition between two uniform flows through which mass and momentum flux is conserved. Although no loss of momentum due to the frictional forces at the bottom is considered, the solution shows that energy must be lost at the bore; and it was suggested that this is due to frictional dissipation or turbulence. If the velocity and depth upstream are u_1 and h_1, and those downstream are u_2 and h_2, then Rayleigh showed that the loss of energy per unit span per unit time was

$$\tfrac{1}{4}\rho g Q \frac{(h_2-h_1)^3}{h_2 h_1}, \tag{1}$$

where

$$Q = u_1 h_1 = u_2 h_2. \tag{2}$$

For a strong bore it is generally accepted that this energy loss occurs by breaking and turbulence just downstream of the bore. But it is found experimentally that weak bores have a stationary train of waves behind them and exhibit no tendency to break (Favre 1935). Lemoine (1948), quoting these results, suggests that, in these circumstances, the required energy loss may occur by radiation through the wave train. Lemoine assumed that the waves are sinusoidal, and calculated their amplitude and resulting rate of radiation of energy through them.

However, his results were only in moderate agreement with experiment. This led Benjamin & Lighthill (1954) to doubt that the wave train is sinusoidal. They decided to follow the suggestion of Keulegan & Patterson (1940), who had shown that the observed waves in Favre's experiments could be taken as sinusoidal waves to a very good approximation. The investigation of Benjamin &

Lighthill (1954) showed that it was possible to match a steady train of waves downstream to a uniform upstream flow only if there was a change in either Q, the volume flow rate, R the energy, or S the momentum flow rate.

For cases where Q and S remain constant they related the resulting wave train behind the bore to the amount by which R was reduced at the bore, varying from waves of very large wavelength, for very little loss of energy, to waves of very small wavelength, when the loss of energy was that given by the classical theory.

In 1965 Favre's results were again analyzed, this time by Sturtevant, who, using the results of the theory of Benjamin & Lighthill, calculated the values of Q, R and S for particular flows. In comparing the values of R and S at the bore with their values upstream, he found that both actually increased at the bore. He went on to show that, in the experiments conducted by Favre, the increase was always several times larger than the decrease predicted by the classical bore conditions. He pointed out, however, that this result, while contrary to the accepted model of the bore (where there is some decrease in S), is not as surprising as it would at first appear. It is in fact consistent with the idea that changes in momentum and energy are due to the action of viscosity near the channel bottom. With co-ordinates fixed in the bore the channel bottom moves faster than the main stream velocity and therefore has the effect of adding momentum and kinetic energy.

In this paper we will look at the effects of laminar viscosity on a bore entering still water, to try to account for the dissipation that Benjamin & Lighthill have shown must occur, in order to obtain a continuous solution for the undular bore.

2. The basic equations

The basic equations for the mean velocity $\mathbf{u}$ and the height h are obtained by the usual procedure of integrating the x-momentum equation across the cross-section of the channel. The additional term due to wall friction appears as a body force in the averaged momentum equations. This method is not new, and is essentially that of Chester (1968) so it will not be repeated in detail here.

The co-ordinate axes are taken so that the x-axis is horizontal and the bottom is $y = 0$. This is shown in figure 1.

For the moment, wall friction will be neglected. The continuity equation and Euler equations are then, in the usual notation,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{3}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{4}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}-g. \tag{5}$$

These are to be solved with a boundary condition at the free surface,

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x} = v. \tag{6}$$

We now integrate (4) from zero to h to obtain

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\frac{\partial \mathbf{u}}{\partial x}+\frac{1}{h}\frac{\partial}{\partial x}\int_0^h (u-\mathbf{u})^2\,dy = -\frac{1}{\rho}\frac{\partial \mathbf{p}}{\partial x}, \tag{7}$$

where

$$\mathbf{L} = \frac{1}{h}\int_0^h L\,dy. \tag{8}$$

The term $(u-\mathbf{u})^2$, appearing on the right-hand side, can be neglected on a cnoidal theory, since it is of order $U_0^2a^4$, where U_0 is a typical velocity scale (say $\surd(gh)$) and a is a typical amplitude of the free-surface disturbance (see Byatt-Smith 1971).

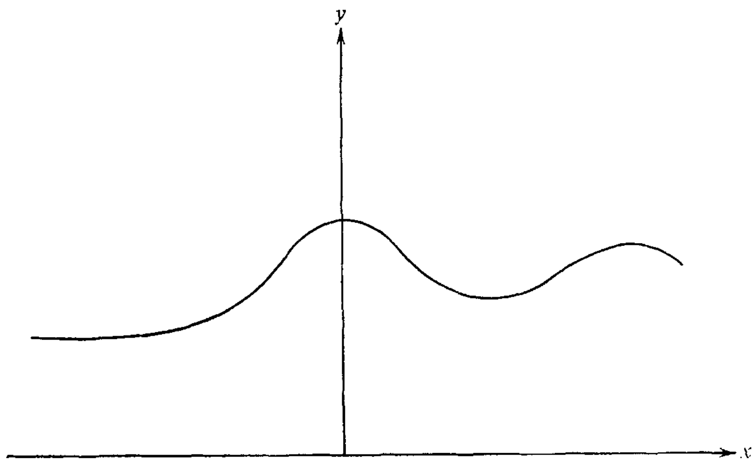

FIGURE 1. The co-ordinate axes.

Also, with the aid of (4), (6) becomes

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(\mathbf{u}h) = 0. \tag{9}$$

The pressure term is evaluated on the basis that the dispersion effect is small and can be accounted for on a two-dimensional linear theory, with dissipation neglected. Again the method follows that of Chester (1968) to obtain the familiar result

$$-\frac{1}{\rho}\frac{\partial \mathbf{p}}{\partial x} = g\frac{\partial h}{\partial x}+gh_0^2\frac{\partial^3 h}{\partial x^3}+O\left(gh_0^4\frac{\partial^5 h}{\partial x^5}\right), \tag{10}$$

where h_0 is the upstream height. Again on a cnoidal theory, we take only the first two terms.

We conclude § 2 with a discussion of the dissipation due to viscous action in the boundary layer on the channel bottom. We assume that the effect of the boundary layer is to add a body force which will appear as a shear stress at the wall. To obtain this we first require the solution of the boundary-layer equations, and for the present purposes it is sufficient to calculate only the most significant terms of the solution. For small disturbances the additional terms can be derived from the linearized boundary-layer equations. Again we follow Chester (1968), who gives a fuller argument. Thus we have to solve

$$\frac{\partial u_b}{\partial t} = \nu \frac{\partial^2 u_b}{\partial y^2}, \tag{11}$$

where u_b is the velocity perturbation in the boundary layer. This gives as solutions

$$u_b = -\tfrac{1}{2} y(\pi\nu)^{-\frac{1}{2}} \int_0^\infty \mathbf{u}(x, t-\zeta) \frac{e^{-y^2}}{4\nu\zeta} \frac{d\zeta}{\zeta^{\frac{3}{2}}}. \tag{12}$$

Finally, we require the effect of the boundary layer on the main flow. From (12) we can obtain the shear stress as

$$\nu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\partial \mathbf{u}}{\partial t}(x, t-\zeta) \frac{d\zeta}{\zeta^{\frac{1}{2}}}. \tag{13}$$

This shear stress will appear on the left-hand side of (7) and the basic equations become

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\mathbf{u}h) = 0, \tag{14}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial \mathbf{u}}{\partial x} + g\frac{\partial h}{\partial x} + \tfrac{1}{3} g h_0^2 \frac{\partial^3 h}{\partial x^3} = -\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\partial \mathbf{u}}{\partial t}(x, t-\zeta) \frac{d\zeta}{\zeta^{\frac{1}{2}}}. \tag{15}$$

3. The effect of viscosity at the channel bottoms

The effect of viscosity is twofold. First, one might expect it to provide some dissipation to account for the loss of energy that is unexplained in the classical theory of the bore. Secondly, one would expect it to have a damping effect on the amplitude of the waves that are found according to an inviscid theory. While these two effects are not uncoupled, the time scales involved are different. For example, an undular bore entering still water develops into a near steady state fairly rapidly, while the amplitude of the leading waves at the head of the bore is damped out very slowly as the waves advance.

We are not really interested in the slow decay of amplitude, so we shall assume that the leading waves are, in fact, steady. Of course, this solution will not be valid at large distances behind the bore, where the boundary layer will not be small. However, it is reasonable to hope that the method of solution will give a fairly good description of the leading waves.

Thus, we look for a steady solution where $\mathbf{u}$ and h are functions of the single variable

$$\xi = x + Ut, \tag{16}$$

where U is the velocity of the bore. Then (14) can be integrated to give

$$(U + \mathbf{u})h = Q = Uh_0, \tag{17}$$

and we can eliminate $\mathbf{u}$ from (14) and (17) to get

$$\left(1 - \frac{U^2 h_0^2}{g h^3}\right) \frac{\partial h}{\partial \xi} + \tfrac{1}{3} h_0^2 \frac{\partial^3 h}{\partial \xi^3} = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \frac{U^2 h_0}{g} \int_0^\infty \left(\frac{1}{h^2} h'\right)_{\xi - U\zeta} \frac{d\zeta}{\zeta^{\frac{1}{2}}}. \tag{18}$$

To non-dimensionalize, we put

$$\xi = h_0 X, \quad \zeta = h_0/U \quad \text{and} \quad h = h_0(1+\eta).$$

Then (18) becomes

$$\tfrac{1}{3}\eta''' - (F^2-1)\,\eta' + 3\eta\eta' = R^{-\frac{1}{2}}\int_0^\infty \eta'(X-\chi)\frac{d\chi}{\chi^{\frac{1}{2}}}, \tag{19}$$

where
$$F^2 = \frac{U^2}{gh_0} \quad \text{and} \quad R = \frac{\pi U h_0}{\nu}. \tag{20}$$

In obtaining this equation from (18) we have made the assumptions that F^2 can be set equal to 1 (except in the term $(F^2-1)\,\eta'$), and that terms of order η^3 can be neglected. This is consistent with the cnoidal theory that we have been using. Also, on the left-hand side we have taken the linear approximation in the integrand. Finally, we can integrate once to obtain

$$\tfrac{1}{3}\eta'' - (F^2-1)\,\eta + \tfrac{3}{2}\eta^2 = R^{-\frac{1}{2}}\int_0^\infty \eta(X-\chi)\frac{d\chi}{\chi^{\frac{1}{2}}}. \tag{21}$$

The constant of integration vanishes, since $\eta \to 0$ as $X \to -\infty$.

4. Discussion of the results and conclusions

Equation (21) is an ordinary integral differential equation; it can be integrated by the following method. The behaviour of η at $-\infty$ is found by looking for a solution of the form $\eta = A\,e^{\alpha}x$, and neglecting the terms of order A^2. We can then integrate (21) by a step-by-step method, using the previous values of η to evaluate the integral. These results are shown in figure 2. When the height of the first trough is plotted against the Froude number, the results, for constant Reynolds number, follow the experimental results of Favre (1935) and Sandover & Zienkiewicz (1957) very well. (See figure 3.) It should be noted, however, that, although the Reynolds number of the experiments of Sandover & Zienkiewicz was higher than that corresponding to the experiments of Favre, the effects of viscosity were larger. This is because the channel used by Sandover & Zienkiewicz was much narrower than that used by Favre, so the effects of the side walls were much more important. Thus, we would expect the theory to be more applicable to the experiments of Favre. However, the theoretical value of the Reynolds number that gives best agreement with Favre's experiments is higher than the actual Reynolds number of the flow. This result is slightly surprising, since it suggests that viscosity is more than able to produce enough dissipation to provide for the loss of energy not accounted for by the classical theory of the bore. In fact, for $F = 1{\cdot}25$ (a strong bore where the amplitude of the leading wave is about $0{\cdot}5$), our theory still gives ample dissipation, provided no breaking occurs to render the theory invalid. Thus it is possible that in a strong bore breaking is not necessary to produce dissipation, but occurs purely because the wave amplitude becomes too large.

It is also of interest to compare the energy and momentum of the wave train behind the bore with the upstream values. To non-dimensionalize we divide the energy by gh_0 and the momentum by gh_0^2. The upstream values are then given by

$$r_u = \tfrac{1}{2}F^2 + 1, \quad s_u = F^2 + \tfrac{1}{2}.$$

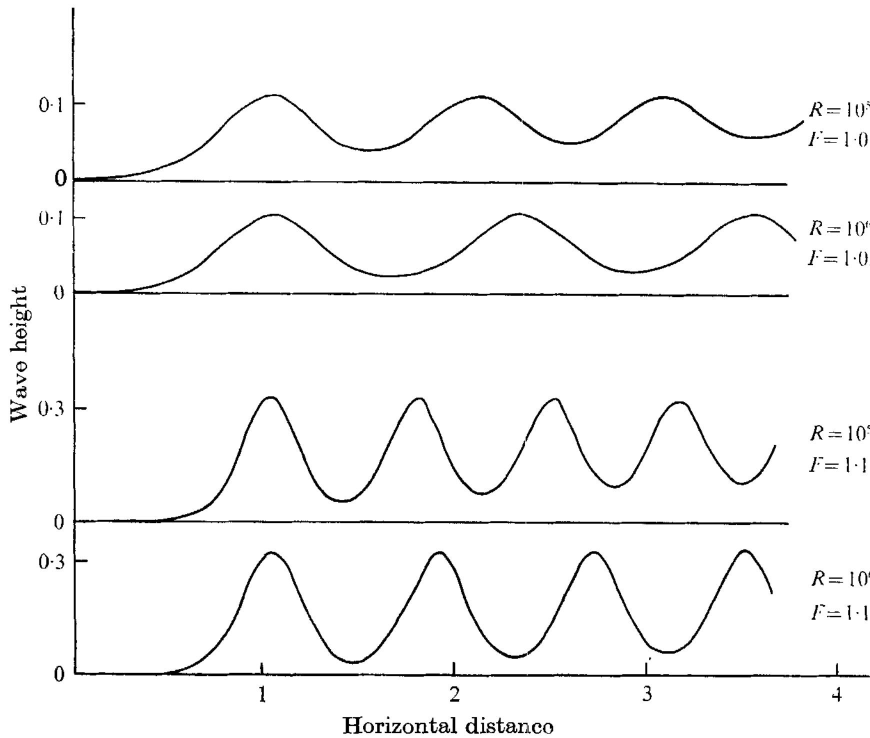

FIGURE 2. A graph showing the effect of viscosity on the solution of the undular bore.

	First trough	Second peak	Second trough	Third peak	Reynolds number	Froude number
$r-r_u$	0·00031	0·00042	0·00050	0·00056	10^{-6}	1·05
$s-s_u$	0·00033	0·00046	0·00055	0·00061		
$r-r_u$	0·0013	0·0016	0·0018	0·0021	10^{-5}	1·05
$s-s_u$	0·0014	0·0017	0·0020	0·0023		
$r-r_u$	0·00059	0·00080	0·00105	0·00122	10^{-6}	1·15
$s-s_u$	0·00075	0·00104	0·00134	0·00168		
$r-r_u$	0·0024	0·0031	0·0038	0·0042	10^{-5}	1·15
$s-s_u$	0·0028	0·0038	0·0049	0·0056		

TABLE 1

The effective charge, $r-r_u$ and $s-s_u$, is then given in table 1. These values are calculated by assuming that the definitions given by Benjamin & Lighthill hold locally over a wavelength; that is, we assume that each wave can be approximated by a cnoidal wave. As we have indicated in § 1, both r and s increase at the bore. However, we notice that the increase continues in the wave train downstream of the bore. This is a consequence of the fact that our dissipation occurs continuously and does not change abruptly at the bore.

It must be remembered, however, that, for obvious theoretical reasons, we have chosen to study a steady profile. Thus we have assumed that no dissipation goes into the damping of the amplitude of the leading waves. From the point of view of comparison, this unsteadiness will not be too important, because large time intervals are required to produce observable changes in profile (Keulegan

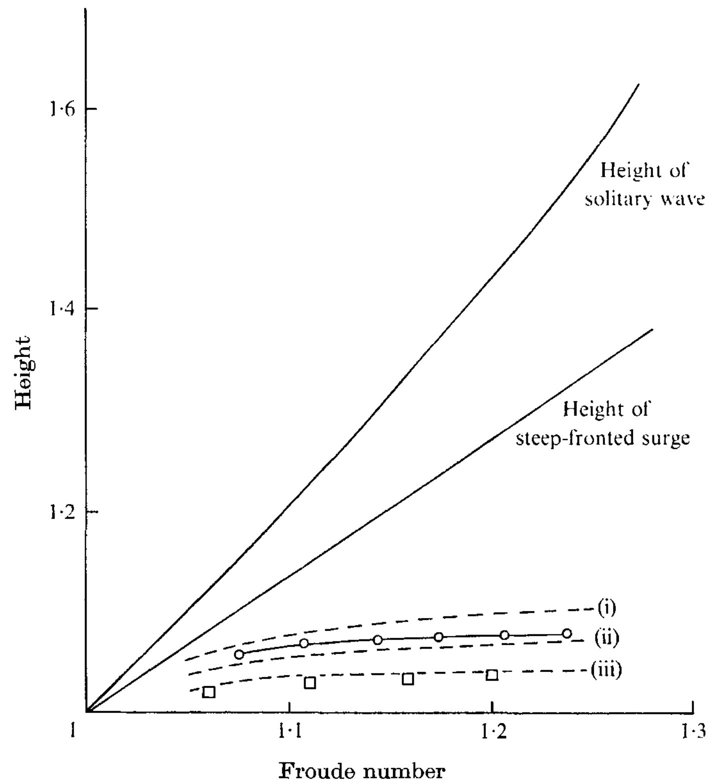

FIGURE 3. A graph showing the effect of viscosity on the height of the first trough. – –, theory: (i) $R = 3\times10^4$, (ii) 10^5, (iii) 10^5. □, Favre (3×10^5); ○, Sandover & Zienkiewicz (9×10^5).

1948). The present analysis does, however, produce a solution which agrees very well with what is observed, and perhaps brings out the mechanism by which this is achieved.

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